Math 155 - Day #26 Expected Values

Suppose that we run a casino and decide to create a gambling game where a contestant rolls one die and wins the number of dollars equal to the amount they rolled. i.e. If they roll a 5 then they win $^{\$}5$. How much should we charge a contestant to play the game?

In choosing price, we want to make sure that we aren't losing money!

But we also can't make it too high, otherwise no one will play

To find a price that is just right, we need to know how much we expect to pay out each time someone plays on average.

This is called the *Expected Value*.

If the game were played 600 times, how times would we expect a 1 to be rolled? Roughly 100 times, since the probability of rolling a 1 is $\frac{1}{6}$

So, in 600 games, we'd expect to pay $^{\$}1 \times 100 =^{\$} 100$ for rolls of 1.

Similarly, we'd expected to pay $^{\$}2 \times 100 =^{\$} 200$ for rolls of 2, $^{\$}3 \times 100 =^{\$} 300$ for rolls of 3, ... $^{\$}6 \times 100 =^{\$} 600$ for rolls of 6.

So, the expected total payout of the game would be:

$$^{\$}1 \times 100 +^{\$} 2 \times 100 +^{\$} 3 \times 100 +^{\$} 4 \times 100 +^{\$} 5 \times 100 +^{\$} 6 \times 100 =^{\$} 2100$$

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Since 600 games were played, we get the expected average per game by dividing the total payout by the number of games played:

$$\frac{1 \cdot 100 + 2 \cdot 100 + 3 \cdot 100 + 4 \cdot 100 + 5 \cdot 100 + 6 \times 100}{600} = \frac{2100}{600} = 3.5$$

On the right side, we find that the expected average payout is $^{\$}3.50$
If we study the left side we can find a quicker way to compute this. Splitting up the fraction on the left, we get:

$$\frac{1 \cdot 100}{600} + \frac{2 \cdot 100}{600} + \frac{3 \cdot 100}{600} + \frac{4 \cdot 100}{600} + \frac{5 \cdot 100}{600} + \frac{6 \cdot 100}{600}$$

$$= 1 \cdot \frac{100}{600} + 2 \cdot \frac{100}{600} + 3 \cdot \frac{100}{600} + 4 \cdot \frac{100}{600} + 5 \cdot \frac{100}{600} + 6 \cdot \frac{100}{600}$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

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So, we re-wrote the expected value (E) of the game to be:

$$E = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

What is the significance of each of those $\frac{1}{6}$?
It is the original probability of that event occurring!
This gives us a way to compute expected values in general.
In General: The *Expected Value* of a trial can be computed by multiplying the value of each outcome by the probability of that outcome, and adding them up. If there are n outcomes labeled $O_1, O_2, \ldots, O_n$:

$$E = O_1 \times P(O_1) + O_2 \times P(O_2) + \cdots + O_n \times P(O_n)$$

As with our first example, casinos use expected values to create games where (on average) "the house always wins".
Another major, local, industry that uses expected values is insurance companies like Mass Mutual. For car insurance, factors such as past accidents, citations, years of driving, etc. are used to compute expected values of the company's pay out. The insurance rate is then set (competitively) above the expected payout.

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Example: Suppose a carnival game is set up so that a player gets to throw a dart to pop a balloon. The board of balloons has 100 balloons on it. Popping a balloon reveals which prize the player wins. Of the 100 balloons, 70 of them give the player a $^{\$}1$ prize, 20 of them give the player a $^{\$}5$ prize, 8 of them give the player a $^{\$}10$ prize, and the last 2 give the player a $^{\$}50$ prize.

What is the expected value of the player's prize value?

$$\begin{aligned} E &= 1 \times P(1) + 5 \times P(5) + 10 \times P(10) + 50 \times P(50) \\ &= 1 \times \tfrac{70}{100} + 5 \times \tfrac{20}{100} + 10 \times \tfrac{8}{100} + 50 \times \tfrac{2}{100} \\ &= 3.50 \end{aligned}$$