Math 155 - Day #19: Combinations

Permutations: The number of ways to order k things from a set of n is called a permutation and is written as:

$$_nP_k = \frac{n!}{(n-k)!}$$

What if we only want to choose part of a group, but not order them?

Example: Suppose that a town has 5 candidates for City Council and voters can choose 3 of them. How many ways can a voter choose 3 candidates from the set of 5?

If the order does not matter then this is call a *Combination*.

We can write the combinations as $_5C_3$

Let's try to count the number of possibilities by labeling the candidates

A, B, C, D, E

ABC ABD ABE ACD

Note: There's no ACB since it's the same as ABC as a combination

ACE ADE BCD BCE BDE CDE - There are 10 total

So, $_5C_3 = 10$

As we've seen with Permutations, Combinations can get very large

So, we need to find a more strategic way to count them!

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What is the different between a combination and permutation?
A Combination ($_nC_k$) chooses k items from a set of n.
A Permutation ($_nP_k$) chooses k items from a set of n and orders k items
Using the ideas from the Fundamental Counting Principle, since Permutations come from two choices (which k items from a set of n and which order for the k items) we have that:

$_nP_k = \left(^{\#} \text{ ways to choose } k \text{ from } n\right) \times \left(^{\#} \text{ of ways to order } k \text{ items }\right)$

The two things on right are the combinations we're trying to understand and the ordering of k items, which we've already found to be k!
So, we can re-write this equation as: $_nP_k =_n C_k \times k!$
So, we can solve this for $_nC_k$ by dividing by k! to get: $_nC_k = \frac{_nP_k}{k!}$
Substituting $_nP_k = \frac{n!}{(n-k)!}$ in this equation, we are left with our formula:

$$_nC_k = \frac{n!}{(n-k)! \times k!}$$

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$$_nC_k = \frac{n!}{(n-k)!k!}$$

Example: Suppose that you have 15 position players on a baseball team and need to choose 9 of them to be the starting players in a game. Notice here, we are not talking batting order - or anything that orders them. Just if they are in the starting group or not.

So, we need to use a combination of 9 from 15

$_{15}C_9 = 5005$

Observation: Notice that we could have equivalently chosen which 6 players did not start

$_{15}C_6 = \frac{15!}{9!6!} = {}_{15}C_9$

There is symmetry in combinations that $_nC_k = {}_nC_{(n-k)}$

Example: How many combinations of 2 Kings are there in a standard deck of 52 cards with 4 Kings?

In this example, we are choosing 2 Kings from a set of 4.

$_4C_2 = 6$

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Example: How many ways can you be dealt 2 Kings in a 5-card hand from a standard 52-card deck?

Here, we have two things to consider: How many ways can the 2 Kings be chosen from the 4 possible Kings AND how many ways can the other 3 cards be chosen from the 48 non-King cards

$^{\#}$ of ways 2 kings can be chosen from $4 = {}_4C_2 = 6$

$^{\#}$ of ways 3 non-kings can be chosen from $48 = {}_{48}C_3 = 17296$

By the FCP, the product of these two gives the total $^{\#}$ of ways to do both:

$^{\#}$ of 5-card hands with 2 kings $= 6 \times 17296 = 103776$

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Example: How many ways can you choose 5 crayons from a set of 16 to put on a table for a child to draw with?
We need a combination since the order of the 5 crayons does not matter
$_{16}C_5 = 4368$
Example: How many ways can 5 cards be dealt from a 52-card deck?
Order of the 5 cards does not matter so we need a combination
$_{52}C_5 = 2598960$
Example: How many ways can 5 cards be dealt with 3 Diamonds?
Note: There are 13 cards of each suit in a standard deck of cards
Here we need to find both the number of ways to be dealt the diamond and non-diamond cards
The number of ways to get 3 diamonds from the 13 is: $_{13}C_3 = 286$
The number of ways to get 2 non-diamonds from the remaining 39 is: $_{39}C_2 = 741$
The number of way to be dealt a 5 card hand with 3 diamonds is:
$_{13}C_3 \times {}_{39}C_2 = 286 \times 741 = 211926$